

# Horizons and Geodesics of Black Ellipsoids

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## Abstract

We analyze the horizon and geodesic structure of a class of 4D off-diagonal metrics with deformed spherical symmetries, which are exact solutions of the vacuum Einstein equations with anholonomic variables. The maximal analytic extension of the ellipsoid type metrics are constructed and the Penrose diagrams are analyzed with respect to adapted frames. We prove that for small deformations (small eccentricities) there are such metrics that the geodesic behaviour is similar to the Schwarzschild one. We conclude that some vacuum static and stationary ellipsoid configurations [1, 2] may describe black ellipsoid objects.

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## 1 Introduction

Recently, the off-diagonal metrics were considered in a new manner by diagonalizing them with respect to anholonomic frames with associated nonlinear connection structure [1, 2, 3]. There were constructed new classes of exact solutions of Einstein's equations in three (3D), four (4D) and five (5D) dimensions. Such vacuum solutions possess a generic geometric local anisotropy (*e.g.* static black hole and cosmological solutions with ellipsoidal or toroidal symmetry, various soliton-dilaton 2D and 3D configurations in 4D gravity, and wormholes and flux tubes with anisotropic polarizations and/or running constants with different extensions to backgrounds of rotation ellipsoids, elliptic cylinders, bipolar and toroidal symmetry and anisotropy).

A number of off-diagonal metrics were investigated in higher dimensional gravity (see, for instance, the Salam, Strathee, Percacci and Randjbar-Daemi works [4] which showed that including off-diagonal components in higher dimensional metrics is equivalent to including  $U(1)$ ,  $SU(2)$  and  $SU(3)$  gauge fields. There are various generalizations of the Kaluza-Klein gravity when the off-diagonal metrics and their compactifications are considered in order to reduce the vacuum 5D gravity to effective Einstein

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gravity and Abelian or non-Abelian gauge theories. One has also constructed 4D exact solutions of Einstein equations with matter fields and cosmological constants like black torus and black strings induced from some 3D black hole configurations by considering 4D off-diagonal metrics whose curvature scalar splits equivalently into a curvature term for a diagonal metric together with a cosmological constant term and/or a Lagrangian for gauge (electromagnetic) field [5].

For some particular off-diagonal metric ansatz and redefinitions of Lagrangians we can model certain effective (diagonal metric) gravitational and matter fields interactions. However, in general, the vacuum gravitational dynamics can not be associated to any matter field contributions. Our aim is to investigate such off-diagonal vacuum gravitational configurations (defined by anholonomic frames with associated nonlinear connection structure) which describe black hole solutions with deformed horizons, for instance, with a static ellipsoid hypersurface.

In this paper we construct the maximal analytic extension of a class of static metrics with deformed spherical symmetry (containing as particular cases ellipsoid configurations). We analyze the Penrose diagrams and compare the results with those for the Reissner-Nordstrom solution. Then we state the conditions when the geodesic congruence with 'ellipsoid' type symmetry can be reduced to the Schwarzschild configuration. We argue that in this case we may generate some static black ellipsoid solutions which, for corresponding parametrizations of off-diagonal metric coefficients, far away from the horizon, satisfy the asymptotic conditions of the Minkowski space-time.

The paper has the following structure: In Sec. 2 we present the necessary formulas on off-diagonal metrics and anholonomic frames with associated nonlinear connection structure and write the vacuum Einstein equations with anholonomic variables corresponding to a general off-diagonal metric ansatz. In Sec. 3 we define a class of static anholonomic deformations of the Schwarzschild metric to some off-diagonal metrics having their coefficients very similar to the Reissner-Nordstrom but written with respect to adapted frames and defined as vacuum configurations. In Sec. 4, for static small ellipsoid deformations, we construct the maximal analytic extension of such metrics and analyze their horizon structure. Section 5 contains a study of the conditions when the geodesic behaviour of ellipsoidal metrics can be congruent to the Schwarzschild one. Conclusions are given in Sec. 6.

## 2 Off-diagonal Metrics and Anholonomic Frames

Let  $V^{3+1}$  be a 4D pseudo-Riemannian space-time enabled with local coordinates  $u^\alpha = (x^i, y^a)$  where the indices of type  $i, j, k, \dots$  run values 1 and 2 and the indices  $a, b, c, \dots$  take values 3 and 4;  $y^3 = v = \varphi$  and  $y^4 = t$  are considered respectively as the "anisotropic" and time like coordinates (subjected to some constraints). This space-time  $V^{3+1}$  we may be provided with a general anholonomic frame structure of tetrads, or vierbiends,

$$e_\alpha = A_\alpha^\beta(u^\gamma) \partial/\partial u^\beta, \quad (1)$$

satisfying some anholonomy relations

$$e_\alpha e_\beta - e_\beta e_\alpha = W_{\alpha\beta}^\gamma(u^\varepsilon) e_\gamma, \quad (2)$$

where  $W_{\alpha\beta}^\gamma(u^\varepsilon)$  are called the coefficients of anholonomy. One defines a 'holonomic' coordinate frames, for instance, a coordinate frame,  $e_\alpha = \partial_\alpha = \partial/\partial u^\alpha$ , as a set of tetrads which satisfy the holonomy conditions

$$\partial_\alpha \partial_\beta - \partial_\beta \partial_\alpha = 0.$$

The 4D line element

$$ds^2 = g_{\alpha\beta}(x^i, v) du^\alpha du^\beta, \quad (3)$$

is parametrized by an ansatz

$$g_{\alpha\beta} = \begin{bmatrix} g_1 + w_1^2 h_3 + n_1^2 h_4 & w_1 w_2 h_3 + n_1 n_2 h_4 & w_1 h_3 & n_1 h_4 \\ w_1 w_2 h_3 + n_1 n_2 h_4 & g_2 + w_2^2 h_3 + n_2^2 h_4 & w_2 h_3 & n_2 h_4 \\ w_1 h_3 & w_2 h_3 & h_3 & 0 \\ n_1 h_4 & n_2 h_4 & 0 & h_4 \end{bmatrix}, \quad (4)$$

with  $g_i = g_i(x^i)$ ,  $h_a = h_{ai}(x^k, v)$  and  $n_i = n_i(x^k, v)$  being some functions of necessary smoothly class or even singular in some points and finite regions. The coefficients  $g_i$  depend only on "holonomic" variables  $x^i$  but the rest of coefficients may also depend on one "anisotropic" (anholonomic) variable  $y^3 = v$ ; the ansatz does not depend on the time variable  $y^4 = t$ : we shall search for static solutions.

We can diagonalize the line element (3),

$$\delta s^2 = g_1(dx^1)^2 + g_2(dx^2)^2 + h_3(\delta v)^2 + h_4(\delta y^4)^2, \quad (5)$$

with respect to the anholonomic co-frame

$$\delta^\alpha = (d^i = dx^i, \delta^a = dy^a + N_i^a dx^i) = (d^i, \delta v = dv + w_i dx^i, \delta y^4 = dy^4 + n_i dx^i) \quad (6)$$

which is dual to the anholonomic frame

$$\delta_\alpha = (\delta_i = \partial_i - N_i^a \partial_a, \partial_b) = (\delta_i = \partial_i - w_i \partial_3 - n_i \partial_4, \partial_3, \partial_4), \quad (7)$$

where  $\partial_i = \partial/\partial x^i$  and  $\partial_b = \partial/\partial y^b$  are usual partial derivatives. The tetrads  $\delta_\alpha$  and  $\delta^\alpha$  are anholonomic because, in general, they satisfy the anholonomy relations (2) with some non-trivial coefficients,

$$W_{ij}^a = \delta_i N_j^a - \delta_j N_i^a, \quad W_{ia}^b = -W_{ai}^b = \partial_a N_i^b.$$

The anholonomy is induced by the coefficients  $N_i^3 = w_i$  and  $N_i^4 = n_i$  which "elongate" partial derivatives and differentials if we are working with respect to anholonomic frames. This results in a more sophisticate differential and integral calculus (as in the tetradic and/or spinor gravity), but simplifies substantially the tensor computations, because we are dealing with diagonalized metrics.

With respect to the anholonomic frames (7) and (6), there is a linear connection, called the canonical distinguished linear connection, which is similar to the metric connection introduced by the Christoffel symbols in the case of holonomic bases, i. e. being constructed only from the metric components and satisfying the metricity conditions  $D_\alpha g_{\beta\gamma} = 0$ . It is parametrized by the coefficients,  $\Gamma^\alpha_{\beta\gamma} = (L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc})$  with the coefficients

$$\begin{aligned} L^i_{jk} &= \frac{1}{2}g^{in}(\delta_k g_{nj} + \delta_j g_{nk} - \delta_n g_{jk}), \\ L^a_{bk} &= \partial_b N^a_k + \frac{1}{2}h^{ac}(\delta_k h_{bc} - h_{dc}\partial_b N^d_k - h_{db}\partial_c N^d_k), \\ C^i_{jc} &= \frac{1}{2}g^{ik}\partial_c g_{jk}, \quad C^a_{bc} = \frac{1}{2}h^{ad}(\partial_c h_{db} + \partial_b h_{dc} - \partial_d h_{bc}), \end{aligned} \quad (8)$$

computed for the ansatz (4), which induce a linear covariant derivative locally adapted to the nonlinear connection structure (N-connection, see details, for instance, in Refs. [6, 1, 7]). We note that on (pseudo) Riemannian spaces the N-connection is an object completely defined by anholonomic frames, when the coefficients of tetradic transform (1),  $A^\beta_\alpha(u^\gamma)$ , are parametrized explicitly via certain values  $(N^a_i, \delta^j_i, \delta^a_b)$ , where  $\delta^j_i$  and  $\delta^a_b$  are the Kronecker symbols. By straightforward calculations we can compute (see, for instance Ref. [8]) that the coefficients of the Levi Civita metric connection

$$\Gamma^{\nabla}_{\alpha\beta\gamma} = g(\delta_\alpha, \nabla_\gamma \delta_\beta) = g_{\alpha\tau} \Gamma^{\nabla\tau}_{\beta\gamma},$$

associated to a covariant derivative operator  $\nabla$ , satisfying the metricity condition  $\nabla_\gamma g_{\alpha\beta} = 0$  for  $g_{\alpha\beta} = (g_{ij}, h_{ab})$ ,

$$\Gamma^{\nabla}_{\alpha\beta\gamma} = \frac{1}{2} \left[ \delta_\beta g_{\alpha\gamma} + \delta_\gamma g_{\beta\alpha} - \delta_\alpha g_{\gamma\beta} + g_{\alpha\tau} W^\tau_{\gamma\beta} + g_{\beta\tau} W^\tau_{\alpha\gamma} - g_{\gamma\tau} W^\tau_{\beta\alpha} \right], \quad (9)$$

are given with respect to the anholonomic basis (6) by the coefficients

$$\Gamma^{\nabla\tau}_{\beta\gamma} = \left( L^i_{jk}, L^a_{bk}, C^i_{jc} + \frac{1}{2}g^{ik}\Omega^a_{jk}h_{ca}, C^a_{bc} \right), \quad (10)$$

where

$$\Omega^a_{jk} = \delta_k N^a_j - \delta_j N^a_k.$$

We emphasize that the (pseudo) Riemannian space-times admit non-trivial torsion components,

$$T^i_{ja} = -T^i_{aj} = C^i_{ja}, \quad T^a_{jk} = -T^a_{kj} = \Omega^a_{kj}, \quad T^a_{bk} = -T^a_{kb} = \partial_b N^a_k - L^a_{bk}, \quad (11)$$

if off-diagonal metrics and anholonomic frames are introduced into consideration. This is a "pure" anholonomic frame effect: the torsion vanishes for the Levi Civita connection stated with respect to a coordinate frame, but even this metric connection contains some torsion coefficients if it is defined with respect to anholonomic frames (this follows from the  $W$ -terms in (9)). We can conclude that the Einstein theory transforms into an effective Einstein-Cartan theory with anholonomically induced torsion if the

general relativity is formulated with respect to general frame bases (both holonomic and anholonomic).

Another specific property of off-diagonal (pseudo) Riemannian metrics is that they can define different classes of linear connections which satisfy the metricity conditions for a given metric, or inversely, there is a certain class of metrics which satisfy the metricity conditions for a given linear connection. This result was originally obtained by A. Kawaguchi [9] (Details can be found in Ref. [6], see Theorems 5.4 and 5.5 in Chapter III, formulated for vector bundles; here we note that similar proofs hold also on manifolds enabled with anholonomic frames associated to a N-connection structure.)

With respect to anholonomic frames, we can not distinguish the Levi Civita connection as the unique both metric and torsionless one. For instance, both linear connections (8) and (10) contain anholonomically induced torsion coefficients, are compatible with the same metric and transform into the usual Levi Civita coefficients for vanishing N-connection and "pure" holonomic coordinates. This means that to an off-diagonal metric in general relativity one may be associated different covariant differential calculi, all being compatible with the same metric structure (like in the non-commutative geometry, which is not a surprising fact because the anolonomic frames satisfy by definition some non-commutative relations (2)). In such cases we have to select a particular type of connection following some physical or geometrical arguments, or to impose some conditions when there is a single compatible linear connection constructed only from the metric and N-coefficients. We note that if  $\Omega_{jk}^a = 0$  the connections (8) and (10) coincide, i. e.  $\Gamma_{\beta\gamma}^\alpha = \Gamma_{\beta\gamma}^{\nabla\alpha}$ .

The vacuum Einstein equations  $R_\alpha^\beta = 0$  computed for the metric (5) and connection (8) with respect to anholonomic frames (6) and (7) transform into a system of partial differential equations with anholonomic variables [1, 2, 7],

$$R_1^1 = R_2^2 = -\frac{1}{2g_1g_2}[g_2^{\bullet\bullet} - \frac{g_1^\bullet g_2^\bullet}{2g_1} - \frac{(g_2^\bullet)^2}{2g_2} + g_1'' - \frac{g_1' g_2'}{2g_2} - \frac{(g_1')^2}{2g_1}] = 0, \quad (12)$$

$$R_3^3 = R_4^4 = -\frac{\beta}{2h_3h_4} = -\frac{1}{2h_3h_4} \left[ h_4^{**} - h_4^* \left( \ln \sqrt{|h_3h_4|} \right)^* \right] = 0, \quad (13)$$

$$R_{3i} = -w_i \frac{\beta}{2h_4} - \frac{\alpha_i}{2h_4} = 0, \quad (14)$$

$$R_{4i} = -\frac{h_4}{2h_3} [n_i^{**} + \gamma n_i^*] = 0, \quad (15)$$

where

$$\alpha_i = \partial_i h_4^* - h_4^* \partial_i \ln \sqrt{|h_3h_4|}, \quad \gamma = 3h_4^*/2h_4 - h_3^*/h_3, \quad (16)$$

and the partial derivatives are denoted  $g_1^\bullet = \partial g_1 / \partial x^1$ ,  $g_1' = \partial g_1 / \partial x^2$  and  $h_3^* = \partial h_3 / \partial v$ . We additionally impose the condition  $\delta_i N_j^a = \delta_j N_i^a$  in order to have  $\Omega_{jk}^a = 0$  which may be satisfied, for instance, if

$$w_1 = w_1(x^1, v), n_1 = n_1(x^1, v), w_2 = n_2 = 0,$$

or, inversely, if

$$w_1 = n_1 = 0, w_2 = w_2(x^2, v), n_2 = n_2(x^2, v).$$

In this case the canonical connection (8) is equivalent to the Levi Civita connection (10) written with respect to anholonomic frames and containing some non-trivial coefficients of induced torsion (11).

In this paper we shall select a class of static solutions parametrized by the conditions

$$w_1 = w_2 = n_2 = 0. \quad (17)$$

The system of equations (12)–(15) can be integrated in general form [7]. Physical solutions are selected following some additional boundary conditions, imposed types of symmetries, nonlinearities and singular behaviour and compatibility in the locally anisotropic limits with some well known exact solutions.

### 3 Anholonomic Deformations of the Schwarzschild Solution

As a background for our investigations we consider an off-diagonal metric ansatz

$$\begin{aligned} \delta s^2 = & \left[ - \left( 1 - \frac{2m}{r} + \frac{\varepsilon}{r^2} \right)^{-1} dr^2 - r^2 q(r) d\theta^2 \right. \\ & \left. - \eta_3(r, \varphi) r^2 \sin^2 \theta d\varphi^2 + \eta_4(r, \varphi) \left( 1 - \frac{2m}{r} + \frac{\varepsilon}{r^2} \right) \delta t^2 \right] \end{aligned} \quad (18)$$

where the "polarization" functions  $\eta_{3,4}$  are decomposed on a small parameter  $\varepsilon$ ,  $0 < \varepsilon \ll 1$ ,

$$\begin{aligned} \eta_3(r, \varphi) &= \eta_{3[0]}(r, \varphi) + \varepsilon \lambda_3(r, \varphi) + \varepsilon^2 \gamma_3(r, \varphi) + \dots, \\ \eta_4(r, \varphi) &= 1 + \varepsilon \lambda_4(r, \varphi) + \varepsilon^2 \gamma_4(r, \varphi) + \dots, \end{aligned} \quad (19)$$

and

$$\delta t = dt + n_1(r, \varphi) dr$$

for  $n_1 \sim \varepsilon \dots + \varepsilon^2$  terms. The functions  $\eta_{3,4}(r, \varphi)$  and  $n_1(r, \varphi)$  will be found as the metric will define a solution of the vacuum Einstein equations generated by small deformations of the spherical static symmetry on a small positive parameter  $\varepsilon$  (in the limits  $\varepsilon \rightarrow 0$  and  $q, \eta_{3,4} \rightarrow 1$  we have just the Schwarzschild solution for a point particle of mass  $m$ ). The metric (18) is a particular case of a class of exact solutions constructed in [1, 2, 7].

The condition of vanishing of the metric coefficient before  $\delta t^2$

$$\begin{aligned} \eta_4(r, \varphi) \left( 1 - \frac{2m}{r} + \frac{\varepsilon}{r^2} \right) &= 1 - \frac{2m}{r} + \varepsilon \frac{\Phi_4}{r^2} + \varepsilon^2 \Theta_4 = 0, \\ \Phi_4 &= \lambda_4 (r^2 - 2mr) + 1 \\ \Theta_4 &= \gamma_4 \left( 1 - \frac{2m}{r} \right) + \lambda_4, \end{aligned} \quad (20)$$

defines a rotation ellipsoid configuration if

$$\lambda_4 = \left( 1 - \frac{2m}{r} \right)^{-1} \left( \cos \varphi - \frac{1}{r^2} \right)$$

and

$$\gamma_4 = -\lambda_4 \left(1 - \frac{2m}{r}\right)^{-1}.$$

In the first order on  $\varepsilon$  one obtains a zero value for the coefficient before  $\delta t^2$  if

$$r_+ = \frac{2m}{1 + \varepsilon \cos \varphi} = 2m[1 - \varepsilon \cos \varphi], \quad (21)$$

which is the equation for a 3D ellipsoid like hypersurface with a small eccentricity  $\varepsilon$ . In general, we can consider arbitrary pairs of functions  $\lambda_4(r, \theta, \varphi)$  and  $\gamma_4(r, \theta, \varphi)$  (for  $\varphi$ -anisotropies,  $\lambda_4(r, \varphi)$  and  $\gamma_4(r, \varphi)$ ) which may be singular for some values of  $r$ , or on some hypersurfaces  $r = r(\theta, \varphi)$  ( $r = r(\varphi)$ ).

The simplest way to analyze the condition of vanishing of the metric coefficient before  $\delta t^2$  is to choose  $\gamma_4$  and  $\lambda_4$  as to have  $\Theta = 0$ . In this case we find from (20)

$$r_{\pm} = m \pm m \sqrt{1 - \varepsilon \frac{\Phi}{m^2}} = m \left[ 1 \pm \left( 1 - \varepsilon \frac{\Phi_4}{2m^2} \right) \right] \quad (22)$$

where  $\Phi_4(r, \varphi)$  is taken for  $r = 2m$ .

Having introduced a new radial coordinate

$$\xi = \int dr \sqrt{\left| 1 - \frac{2m}{r} + \frac{\varepsilon}{r^2} \right|} \quad (23)$$

and defined

$$h_3 = -\eta_3(\xi, \varphi) r^2(\xi) \sin^2 \theta, \quad h_4 = 1 - \frac{2m}{r} + \varepsilon \frac{\Phi_4}{r^2}, \quad (24)$$

for  $r = r(\xi)$  found as the inverse function after integration in (23), we write the metric (18) as

$$\begin{aligned} ds^2 &= -d\xi^2 - r^2(\xi) q(\xi) d\theta^2 + h_3(\xi, \theta, \varphi) \delta\varphi^2 + h_4(\xi, \theta, \varphi) \delta t^2, \\ \delta t &= dt + n_1(\xi, \varphi) d\xi, \end{aligned} \quad (25)$$

where the coefficient  $n_1$  is taken to solve the equation (15) and to satisfy the (17).

Let us define the conditions when the coefficients of metric (18) define solutions of the vacuum Einstein equations: For  $g_1 = -1$ ,  $g_2 = -r^2(\xi) q(\xi)$  and arbitrary  $h_3(\xi, \theta, \varphi)$  and  $h_4(\xi, \theta)$  we get solutions the equations (12)–(14). If  $h_4$  depends on anisotropic variable  $\varphi$ , the equation (13) may be solved if

$$\sqrt{|\eta_3|} = \eta_0 \left( \sqrt{|\eta_4|} \right)^* \quad (26)$$

for  $\eta_0 = \text{const.}$  Considering decompositions of type (19) we put  $\eta_0 = \eta/\varepsilon$  where the constant  $\eta$  is taken as to have  $\sqrt{|\eta_3|} = 1$  in the limits

$$\frac{\left( \sqrt{|\eta_4|} \right)^* \rightarrow 0}{\varepsilon \rightarrow 0} \rightarrow \frac{1}{\eta} = \text{const.} \quad (27)$$

These conditions are satisfied if the functions  $\eta_{3[0]}$ ,  $\lambda_{3,4}$  and  $\gamma_{3,4}$  are related via relations

$$\sqrt{|\eta_{3[0]}|} = \frac{\eta}{2} \lambda_4^*, \lambda_3 = \eta \sqrt{|\eta_{3[0]}|} \gamma_4^*$$

for arbitrary  $\gamma_3(r, \varphi)$ . In this paper we select only such solutions which satisfy the conditions (26) and (27).

In order to consider linear infinitesimal extensions on  $\varepsilon$  of the Schwarzschild metric we may write the solution of (15) as

$$n_1 = \varepsilon \hat{n}_1(\xi, \varphi)$$

where

$$\begin{aligned} \hat{n}_1(\xi, \varphi) &= n_{1[1]}(\xi) + n_{1[2]}(\xi) \int d\varphi \eta_3(\xi, \varphi) / \left( \sqrt{|\eta_4(\xi, \varphi)|} \right)^3, \eta_4^* \neq 0; \\ &= n_{1[1]}(\xi) + n_{1[2]}(\xi) \int d\varphi \eta_3(\xi, \varphi), \eta_4^* = 0; \\ &= n_{1[1]}(\xi) + n_{1[2]}(\xi) \int d\varphi / \left( \sqrt{|\eta_4(\xi, \varphi)|} \right)^3, \eta_3^* = 0; \end{aligned} \quad (28)$$

with the functions  $n_{k[1,2]}(\xi)$  to be stated by boundary conditions.

The data

$$\begin{aligned} g_1 &= -1, g_2 = -r^2(\xi)q(\xi), \\ h_3 &= -\eta_3(\xi, \varphi)r^2(\xi)\sin^2\theta, \quad h_4 = 1 - \frac{2m}{r} + \varepsilon \frac{\Phi_4}{r^2}, \\ w_{1,2} &= 0, n_1 = \varepsilon \hat{n}_1(\xi, \varphi), n_2 = 0, \end{aligned} \quad (29)$$

for the metric (18) define a class of solutions of the vacuum Einstein equations with non-trivial polarization function  $\eta_3$  and extended on parameter  $\varepsilon$  up to the second order (the polarization functions being taken as to make zero the second order coefficients). Such solutions are generated by small deformations (in particular cases of rotation ellipsoid symmetry) of the Schwarzschild metric.

We can relate our solutions with some small deformations of the Schwarzschild metric, as well we can satisfy the asymptotically flat condition, if we chose such functions  $n_{k[1,2]}(x^i)$  as  $n_k \rightarrow 0$  for  $\varepsilon \rightarrow 0$  and  $\eta_3 \rightarrow 1$ . These functions have to be selected as to vanish far away from the horizon, for instance, like  $\sim 1/r^{1+\tau}$ ,  $\tau > 0$ , for long distances  $r \rightarrow \infty$ .

## 4 Analytic Extensions of Ellipsoid Metrics

The metric (18) (equivalently (25)) considered with respect to the anholonomic frame (6) has a number of similarities with the Schwarzschild and Reissner–Nördstrom solutions. If we identify  $\varepsilon$  with  $e^2$ , we get a static metric with effective "electric" charge induced by a small, quadratic on  $\varepsilon$ , off-diagonal metric extension. The coefficients of this metric are similar to those from the Reissner–Nördstrom solution but additionally



to the mentioned frame anholonomy there are additional polarizations by the functions  $h_{3[0]}$ ,  $\eta_{3,4}$  and  $n_1$ . Another very important property is that the deformed metric was stated to define a vacuum solution of the Einstein equations which differs substantially from the usual Reissner–Nördstrom metric being an exact static solution of the Einstein–Maxwell equations.

For  $\varepsilon \rightarrow 0$  and  $h_{3[0]} \rightarrow 1$  the metric (18) transforms into the usual Schwarzschild metric. A solution with ellipsoid symmetry can be selected by a corresponding condition of vanishing of the coefficient before  $\delta t$  which defines an ellipsoidal hypersurface like for the Kerr metric, but in our case the metric is non-rotating.

The metric (18) has a singular behaviour for  $r = r_{\pm}$ , see (22). The aim of this section is to prove that this way we have constructed a solution of the vacuum Einstein equations with an "anisotropic" horizon being a small deformation on parameter  $\varepsilon$  of the Schwarzschild's solution horizon. We may analyze the anisotropic horizon's properties for some fixed "direction" given in a smooth vicinity of any values  $\varphi = \varphi_0$  and  $r_+ = r_+(\varphi_0)$ . The final conclusions will be some general ones for arbitrary  $\varphi$  when the explicit values of coefficients will have a parametric dependence on angular coordinate  $\varphi$ . The metrics (18) and (25) are regular in the regions I ( $\infty > r > r_+^{\Phi}$ ), II ( $r_+^{\Phi} > r > r_-^{\Phi}$ ) and III ( $r_-^{\Phi} > r > 0$ ). As in the Schwarzschild, Reissner–Nördstrom and Kerr cases these singularities can be removed by introducing suitable coordinates and extending the manifold to obtain a maximal analytic extension [12, 13]. We have similar regions as in the Reissner–Nördstrom space-time, but with just only one possibility  $\varepsilon < 1$  instead of three relations for static electro-vacuum cases ( $e^2 < m^2, e^2 = m^2, e^2 > m^2$ ; where  $e$  and  $m$  are correspondingly the electric charge and mass of the point particle in the Reissner–Nördstrom metric). So, we may consider the usual Penrose's diagrams as for a particular case of the Reissner–Nördstrom space-time but keeping in mind that such diagrams and horizons have an additional parametrization on an angular coordinate.

To construct the maximally extended manifold, we proceed in steps analogous to those in the Schwarzschild case (see details, for instance, in Ref. [10]). We introduce a new coordinate

$$r^{\parallel} = \int dr \left( 1 - \frac{2m}{r} + \frac{\varepsilon}{r^2} \right)^{-1}$$

for  $r > r_+^1$  and find explicitly the coordinate

$$r^{\parallel} = r + \frac{(r_+^1)^2}{r_+^1 - r_-^1} \ln(r - r_+^1) - \frac{(r_-^1)^2}{r_+^1 - r_-^1} \ln(r - r_-^1), \quad (30)$$

where  $r_{\pm}^1 = r_{\pm}^{\Phi}$  with  $\Phi = 1$ . If  $r$  is expressed as a function on  $\xi$ , than  $r^{\parallel}$  can be also expressed as a function on  $\xi$  depending additionally on some parameters.

Defining the advanced and retarded coordinates,  $v = t + r^{\parallel}$  and  $w = t - r^{\parallel}$ , with corresponding elongated differentials

$$\delta v = \delta t + dr^{\parallel} \text{ and } \delta w = \delta t - dr^{\parallel}$$

the metric (25) takes the form

$$\delta s^2 = -r^2(\xi)q(\xi)d\theta^2 - \eta_3(\xi, \varphi_0)r^2(\xi)\sin^2\theta\delta\varphi^2 + \left(1 - \frac{2m}{r(\xi)} + \varepsilon\frac{\Phi_4(r, \varphi_0)}{r^2(\xi)}\right)\delta v\delta w,$$

where (in general, in non-explicit form)  $r(\xi)$  is a function of type  $r(\xi) = r(r^\parallel) = r(v, w)$ . Introducing new coordinates  $(v'', w'')$  by

$$v'' = \arctan \left[ \exp \left( \frac{r_+^1 - r_-^1}{4(r_+^1)^2} v \right) \right], w'' = \arctan \left[ -\exp \left( \frac{-r_+^1 + r_-^1}{4(r_+^1)^2} w \right) \right]$$

and multiplying the last term on the conformal factor we obtain

$$\delta s^2 = -r^2 q(r) d\theta^2 - \eta_3(r, \varphi_0) r^2 \sin^2 \theta \delta \varphi^2 + 64 \frac{(r_+^1)^4}{(r_+^1 - r_-^1)^2} \left( 1 - \frac{2m}{r(\xi)} + \varepsilon \frac{\Phi_4(r, \varphi_0)}{r^2(\xi)} \right) \delta v'' \delta w'', \quad (31)$$

where  $r$  is defined implicitly by

$$\tan v'' \tan w'' = -\exp \left[ \frac{r_+^1 - r_-^1}{2(r_+^1)^2} r \right] \sqrt{\frac{r - r_+^1}{(r - r_-^1)^\chi}}, \chi = \left( \frac{r_+^1}{r_-^1} \right)^2.$$

As particular cases, we may chose  $\eta_3(r, \varphi)$  as the condition of vanishing of the metric coefficient before  $\delta v'' \delta w''$  will describe a horizon parametrized by a resolution ellipsoid hypersurface.

The maximal extension of the Schwarzschild metric deformed by a small parameter  $\varepsilon$  (for ellipsoid configurations treated as the eccentricity), i. e. the extension of the metric (18), is defined by taking (31) as the metric on the maximal manifold on which this metric is of smoothly class  $C^2$ . The Penrose diagram of this static but locally anisotropic space-time, for any fixed angular value  $\varphi_0$  is similar to the Reissner-Nordstrom solution, for the case  $e^2 \rightarrow \varepsilon$  and  $e^2 < m^2$  (see, for instance, Ref. [10]). There are an infinite number of asymptotically flat regions of type I, connected by intermediate regions II and III, where there is still an irremovable singularity at  $r = 0$  for every region III. We may travel from a region I to another ones by passing through the 'wormholes' made by anisotropic deformations (ellipsoid off-diagonality of metrics, or anholonomy) like in the Reissner-Nordstrom universe because  $\sqrt{\varepsilon}$  may model an effective electric charge. One can not turn back in a such travel.

It should be noted that the metric (31) is analytic every were except at  $r = r_-^1$ . We may eliminate this coordinate degeneration by introducing another new coordinates

$$v''' = \arctan \left[ \exp \left( \frac{r_+^1 - r_-^1}{2n_0(r_+^1)^2} v \right) \right], w''' = \arctan \left[ -\exp \left( \frac{-r_+^1 + r_-^1}{2n_0(r_+^1)^2} w \right) \right],$$

where the integer  $n_0 \geq (r_+^1)^2 / (r_-^1)^2$ . In these coordinates, the metric is analytic every were except at  $r = r_+^1$  where it is degenerate. This way the space-time manifold can be covered by an analytic atlas by using coordinate carts defined by  $(v'', w'', \theta, \varphi)$  and  $(v''', w''', \theta, \varphi)$ . Finally we note that the analytic extensions of the deformed metrics were performed with respect to anholonomic frames which distinguish such constructions from those dealing only with holonomic coordinates, like for the usual Reissner-Nördstrom and Kerr metrics.

## 5 Geodesics on Static Ellipsoid Backgrounds

In this section we analyze the geodesic congruence of the metric (25) with the data (29), for simplicity, being linear on  $\varepsilon$ , by introducing the effective Lagrangian (for instance, like in Ref. [14])

$$2L = g_{\alpha\beta} \frac{\delta u^\alpha}{ds} \frac{\delta u^\beta}{ds} = - \left(1 - \frac{2m}{r} + \frac{\varepsilon}{r^2}\right)^{-1} \left(\frac{dr}{ds}\right)^2 - r^2 q(r) \left(\frac{d\theta}{ds}\right)^2 - \eta_3(r, \varphi) r^2 \sin^2 \theta \left(\frac{d\varphi}{ds}\right)^2 + \left(1 - \frac{2m}{r} + \frac{\varepsilon \Phi_4}{r^2}\right) \left(\frac{dt}{ds} + \varepsilon \hat{n}_1 \frac{dr}{ds}\right)^2, \quad (32)$$

for  $r = r(\xi)$ .

The corresponding Euler–Lagrange equations,

$$\frac{d}{ds} \frac{\partial L}{\partial \frac{\delta u^\alpha}{ds}} - \frac{\partial L}{\partial u^\alpha} = 0$$

are

$$\begin{aligned} \frac{d}{ds} \left[ -r^2 q(r) \frac{d\theta}{ds} \right] &= -\eta_3 r^2 \sin \theta \cos \theta \left( \frac{d\varphi}{ds} \right)^2, \\ \frac{d}{ds} \left[ -\eta_3 r^2 \frac{d\varphi}{ds} \right] &= -\eta_3^* \frac{r^2}{2} \sin^2 \theta \left( \frac{d\varphi}{ds} \right)^2 + \frac{\varepsilon}{2} \left( 1 - \frac{2m}{r} \right) \left[ \frac{\Phi_4^*}{r^2} \left( \frac{dt}{ds} \right)^2 + \hat{n}_1^* \frac{dt}{ds} \frac{d\xi}{ds} \right] \end{aligned} \quad (33)$$

and

$$\frac{d}{ds} \left[ \left( 1 - \frac{2m}{r} + \frac{\varepsilon \Phi_4}{r^2} \right) \left( \frac{dt}{ds} + \varepsilon \hat{n}_1 \frac{d\xi}{ds} \right) \right] = 0, \quad (34)$$

where, for instance,  $\Phi_4^* = \partial \Phi_4 / \partial \varphi$  we have omitted the variations for  $d\xi/ds$  which may be found from (32). The sistem of equations (32)–(34) transform into the usual system of geodesic equations for the Schwarzschild space–time if  $\varepsilon \rightarrow 0$  and  $q, \eta_3 \rightarrow 1$  which can be solved exactly [14]. For nontrivial values of the parameter  $\varepsilon$  and polarization  $\eta_3$  even to obtain some decompositions of solutions on  $\varepsilon$  for arbitrary  $\eta_3$  and  $n_{1[1,2]}$ , see (28), is a cumbersome task. In spite of the fact that with respect to anholonomic frames the metric (25) has the coefficients being very similar to the Reissner–Nordstrom solution the geodesic behaviour, in our anisotropic case, is more sophisticate because of anholonomy and "elongation" of partial derivatives. For instance, the equations (33) state that there is not any angular on  $\varphi$ , conservation law if  $\eta_3^* \neq 0$ , even for  $\varepsilon \rightarrow 0$  (which holds both for the Schwarzschild and Reissner–Nordstrom metrics). One follows from the equation (34) the existence of an energy like integral of motion,  $E = E_0 + \varepsilon E_1$ , with

$$\begin{aligned} E_0 &= \left( 1 - \frac{2m}{r} \right) \frac{dt}{ds} \\ E_1 &= \frac{\Phi_4}{r^2} \frac{dt}{ds} + \left( 1 - \frac{2m}{r} \right) \hat{n}_1 \frac{d\xi}{ds}. \end{aligned}$$

The introduced anisotropic deformations of congruences of Schwarzschild's space–time geodesics maintain the known behaviour in the vicinity of the horizon hypersurface

defined by the condition of vanishing of the coefficient  $(1 - 2m/r + \varepsilon\Phi_4/r^2)$  in (31). The simplest way to prove this is to consider radial null geodesics in the "equatorial plane", which satisfy the condition (32) with  $\theta = \pi/2, d\theta/ds = 0, d^2\theta/ds^2 = 0$  and  $d\varphi/ds = 0$ , from which follows that

$$\frac{dr}{dt} = \pm \left(1 - \frac{2m}{r} + \frac{\varepsilon_0}{r^2}\right) [1 + \varepsilon\hat{n}_1 d\varphi].$$

The integral of this equation, for every fixed value  $\varphi = \varphi_0$  is

$$t = \pm r^{\parallel} + \varepsilon \int \left[ \frac{\Phi_4(r, \varphi_0) - 1}{2(r^2 - 2mr)^2} - \hat{n}_1(r, \varphi_0) \right] dr$$

where the coordinate  $r^{\parallel}$  is defined in equation (30). In this formula the term proportional to  $\varepsilon$  can have non-singular behaviour for a corresponding class of polarizations  $\lambda_4$ , see the formulas (20). Even the explicit form of the integral depends on the type of polarizations  $\eta_3(r, \varphi_0)$  and  $n_{1[1,2]}(r)$ , which results in some small deviations of the null-geodesics, we may conclude that for an in-going null-ray the coordinate time  $t$  increases from  $-\infty$  to  $+\infty$  as  $r$  decreases from  $+\infty$  to  $r_+^1$ , decreases from  $+\infty$  to  $-\infty$  as  $r$  further decreases from  $r_+^1$  to  $r_-^1$ , and increases again from  $-\infty$  to a finite limit as  $r$  decreases from  $r_-^1$  to zero. We have a similar behaviour as for the Reissner-Nordstrom solution but with some additional anisotropic contributions being proportional to  $\varepsilon$ . Here we also note that as  $dt/ds$  tends to  $+\infty$  for  $r \rightarrow r_+^1 + 0$  and to  $-\infty$  as  $r_- + 0$ , any radiation received from infinity appear to be infinitely red-shifted at the crossing of the event horizon and infinitely blue-shifted at the crossing of the Cauchy horizon.

The mentioned properties of null-geodesics allow us to conclude that the metric (18) (equivalently, (25)) with the data (29) and their maximal analytic extension (31) really define a black hole static solution which is obtained by anisotropic small deformations on  $\varepsilon$  and renormalization by  $\eta_3$  of the Schwarzschild solution (for a corresponding type of deformations the horizon of such black holes is defined by ellipsoid hypersurfaces). We call such objects as black ellipsoids, or black rotoids. They exist in the framework of general relativity as certain vacuum solutions of the Einstein equations defined by static generic off-diagonal metrics and associated anholonomic frames. This property distinguishes them from similar configurations of Reissner-Norstrom type (which are static electrovacuum solutions of the Einstein-Maxwell equations) and of Kerr type rotating configurations, with ellipsoid horizon, also defined by off-diagonal vacuum metrics (here we emphasized that the spherical coordinate system is associated to a holonomic frame which is a trivial case of anholonomic bases).

## 6 Conclusions

We proved that there are such small, with nonlinear gravitational polarization, static deformations of the Schwarzschild black hole solution (for instance, to some resolution ellipsoid like configurations) which preserve the horizon and geodesic behaviour, but slightly deform the spherical constructions. This means that we may state such parameters of the exact solutions of vacuum Einstein equations defined by off-diagonal

metrics with ellipsoid symmetry, constructed in Refs. [1, 2, 7], as the solutions would define static ellipsoid black hole configurations.

The new class of static ellipsoidal black hole metrics possesses a number of similarities with the Reissner–Nordstrom metric: The parameter of ellipsoidal deformation may be considered as an effective electromagnetic charge induced by off-diagonal vacuum gravitational interactions. Effective electromagnetic charges and Reissner–Nordstrom metrics, induced by interactions in the bulk of extra dimension gravity, were considered recently in brane gravity [16]. In this paper (see also Refs. [7]) we proved that such Reissner–Nordstrom like ellipsoid black hole configurations may be constructed even in the framework of vacuum Einstein gravity if off-diagonal metrics and anholonomic frames are introduced into consideration.

We emphasize that the static ellipsoid black holes possess spherical topology and satisfy the principle of topological censorship [15]. They are also compatible with the black hole uniqueness theorems [17]; at asymptotics, at least for a very small eccentricity, such metrics transform into the usual Schwarzschild one. We note that the stability of static ellipsoid black holes can be proved similarly by considering small perturbations of the spherical black holes [18]. (On the stability of the Schwarzschild solution see details in Ref. [14].)

It is interesting to compare the off-diagonal ellipsoidal metrics with those describing the distorted diagonal black hole solutions (see the vacuum case in Refs. [19] and an extension to the case of non-vanishing electric fields [20]). In our ellipsoidal cases the distortion of spacetime is caused by some anisotropic off-diagonal terms being non-trivial in some regions but in the case of "pure diagonal" distortions one treats such effects following the fact that the vacuum Einstein equations are not satisfied in some regions because of presence of matter. As we emphasized in the introduction section, the off-diagonal gravity may model some gravity–matter like interactions (like in Kaluza–Klein theory, for some very particular configurations and topological compactifications) but, in general, the off-diagonal vacuum gravitational dynamics can not be associated to any effective matter dynamics in a holonomic gravitational background. So, we may consider that the anholonomic ellipsoidal deformations of the Schwarzschild metric are some kind of anisotropic off-diagonal distortions modeled by certain vacuum gravitational fields with the distortion parameters (equivalently, vacuum gravitational polarizations) depending both on radial and angular coordinates. In general, both classes of off-diagonal anisotropic and "pure" diagonal distortions (like in Refs. [19]) result in solutions which are not asymptotically flat. However, it is possible to find asymptotically flat extensions, as it was shown in this paper, even for ellipsoidal configurations by introducing the corresponding off-diagonal terms (the asymptotic conditions for the diagonal distortions are discussed in Ref. [20]. To satisfy such conditions one has to include some additional matter fields in the exterior portion of spacetime.)

Using the methods elaborated and developed in Refs. [1, 7, 3], and in this paper, we can construct off-diagonal ellipsoidal extensions of the already diagonally distorted Schwarzschild metric (see the metric (3.7) from Ref. [20]). Such anholonomic deformations would contain in the diagonal limit configurations with  $\varepsilon \rightarrow 0$  but  $\eta_3 \neq 1$ ; for such configurations the function  $\eta_3$  has to be related in the corresponding limits with the

values  $\overline{\gamma}_D, \overline{\psi}_D$  and  $A$  from [20]. We remark that there are different classes of ellipsoidal deformations of the Schwarzschild metric which result in a vacuum configuration. The conditions  $\varepsilon \rightarrow 0$  and  $q, \eta_3 = 1$  select just the limit of the usual radial Schwarzschild asymptotics without any (also possible) additional diagonal distortions.

The deformation parameter  $\varepsilon$  effectively seems to put an "electric charge" on the black hole which is of gravitational off-diagonal/anhologonomic origin. It can describe effectively both positive and negative gravitational polarizations (even some repulsive gravitational effects). This is not surprising because the coefficients of an anisotropic black hole are similar to those of the Reissner–Nordstrom solution only with respect to corresponding anholonomic frames which are subjected to some constraints (anholonomy conditions). Intuitively, we may compare such effects with those from the usual Newton gravity: a ball falls directly on the Earth but it can run under an angle inside a tube because of constraints imposed at the boundary (by anholonomic frames in general relativity we may model a more sophisticated behaviour with locally anisotropic gravitational polarizations and even repulsion).

For the ellipsoidal metrics with the Schwarzschild asymptotics, the ellipsoidal character could result in some observational effects in the vicinity of the horizon (for instance, scattering of particles on a static ellipsoid; we can compute anisotropic matter accretion effects on an ellipsoidal black hole put in the center of a galactic being of ellipsoidal or another configuration). A point of further investigations could be the anisotropic ellipsoidal collapse when both the matter and spacetime are of ellipsoidal generic off-diagonal symmetry (former theoretical and computational investigations were performed only for rotoids with anisotropic matter and particular classes of perturbations of the Schwarzschild solutions [21]). For very small eccentricities, we may not have any observable effects like perihelion shift or light bending if we restrict our investigations only to the Schwarzschild–Newton asymptotics.

Finally, we present some comments on mechanics and thermodynamics of ellipsoidal black holes. For the static black ellipsoids with flat asymptotics, we can compute the area of the ellipsoidal horizon, associate an entropy and develop a corresponding black ellipsoid thermodynamics. But this would be a very rough approximation because, in general, we are dealing with off-diagonal metrics depending anisotropically on two/three coordinates. Such solutions are with anholonomically deformed Killing horizons and should be described by a thermodynamics (in general, both non-equilibrium and irreversible) of black ellipsoids self-consistently embedded into an off-diagonal anisotropic gravitational vacuum. This is a ground for numerous new conceptual issues to be developed and related to anisotropic black holes and the anisotropic kinetics and thermodynamics [2] as well to a framework of isolated anisotropic horizons [22] which is a matter of our further investigations.

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